Physics 566: Quantum Optics I

Problem Set 1, Solutions

Problem 1: The Weiner-Khintchine Theorm

An important result of classical coherence theory is the relationship between an auto-correlation function and the spectrum of the signal.

Let $K(t_2-t_1) = \langle f(t_1) f(t_2)\rangle = \langle f(0) f(t_2-t_1)\rangle$ = Auto correlation function for a real stationary signal (correlation depends only on the time-difference t_2-t_1.

The $\langle\ \rangle$ is taken as the ensemble average over the probability distribution over $f(t)$'s.

(a) Consider the correlation of the Fourier transform of the signal at frequencies ω and ω'.

$$\langle \tilde{f}^*(\omega)\, \tilde{f}(\omega')\rangle = \left\langle \left(\int_{-\infty}^{\infty} dt\ f(t)\, e^{+i\omega t}\right)^* \left(\int_{-\infty}^{\infty} dt'\, f(t')\, e^{+i\omega' t'}\right)\right\rangle$$

$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\ e^{i(\omega' t' - \omega t)}\ \underbrace{\langle f(t) f(t')\rangle}_{K(t'-t)}$$

Change variables: $\tau = t'-t$, $T = \dfrac{t+t'}{2}$, Jacobian $dt_1 dt_2 = dt' dT$

$$\Rightarrow \langle \tilde{f}^*(\omega)\, \tilde{f}(\omega')\rangle = \int_{-\infty}^{\infty} dT\ e^{i(\omega'-\omega)T} \int_{-\infty}^{\infty} d\tau\ e^{i(\omega+\omega')\frac{\tau}{2}} K(\tau) = 2\pi\,\delta(\omega-\omega') \int_{-\infty}^{\infty} e^{i\omega\tau} K(\tau)\, d\tau$$

$$\boxed{\Rightarrow \langle \tilde{f}^*(\omega)\, \tilde{f}(\omega')\rangle = 2\pi\, S(\omega)\, \delta(\omega-\omega'), \qquad S(\omega) = \int_{-\infty}^{\infty} d\tau\ e^{i\omega\tau} \langle f(0) f(\tau)\rangle \equiv \text{Spectral density}}$$

(b) Consider now the correlation between the complex "analytic signals" (equals twice the positive frequency component) of the electric field: $E(t) \equiv \mathrm{Re}(\underbrace{\tilde{E}(t)}_{\text{analytic signal}}) = \frac{1}{2}(\tilde{E}(t) + \tilde{E}^*(t))$.

$\Gamma(\tau) \equiv \langle \tilde{E}(0)\, \tilde{E}^*(\tau)\rangle$

Fourier transform of the real signal $\tilde{E}(\omega) = \int_{-\infty}^{\infty} dt\ E(t)\, e^{-i\omega t}$

Analytic signal: $\tilde{E}(t) = 2E^{(+)}(t) = 2\int_0^{\infty} \frac{d\omega}{2\pi}\ \tilde{E}(\omega)\, e^{-i\omega t}$

Consider $\frac{1}{2}\mathrm{Re}[\Gamma(\tau)] = \frac{1}{4}\left[\langle \tilde{E}^*(0)\tilde{E}(\tau)\rangle + \langle \tilde{E}^*(\tau)\tilde{E}(0)\rangle\right] = \int_0^\infty \frac{d\omega}{2\pi}\int_0^\infty \frac{d\omega'}{2\pi}\langle \tilde{E}^*(\omega)\tilde{E}(\omega')\rangle e^{-i\omega\tau} + c.c.$

$$= \int_0^\infty \frac{d\omega}{2\pi}\int_0^\infty \frac{d\omega'}{2\pi} 2\pi S(\omega)\,\delta(\omega-\omega')\, e^{-i\omega\tau} + c.c. = \int_0^\infty \frac{d\omega}{2\pi} S(\omega) e^{-i\omega\tau} + \int_0^\infty \frac{d\omega}{2\pi} S^*(\omega) e^{i\omega\tau}.$$

Aside: $S^*(\omega) = \left[\int_{-\infty}^{\infty} d\tau\, K(\tau)\, e^{+i\omega\tau}\right]^* = \int_{-\infty}^{\infty} d\tau\, K(\tau)\, e^{-i\omega\tau} = S(-\omega)$

$$\Rightarrow \boxed{\frac{1}{2}\mathrm{Re}[\Gamma(\tau)] = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi} S(\omega)\, e^{-i\omega\tau} \Rightarrow S(\omega) = \int_{-\infty}^{\infty} d\tau\, \frac{1}{2}\mathrm{Re}[\Gamma(\tau)]\, e^{+i\omega\tau}}$$

(c) The power spectrum is defined by the spectral density $S(\omega)$. Consider "natural light" e.g. light from a thermal lamp, collision broadened. We found in class, the autocorrelation function between the complex amplitudes

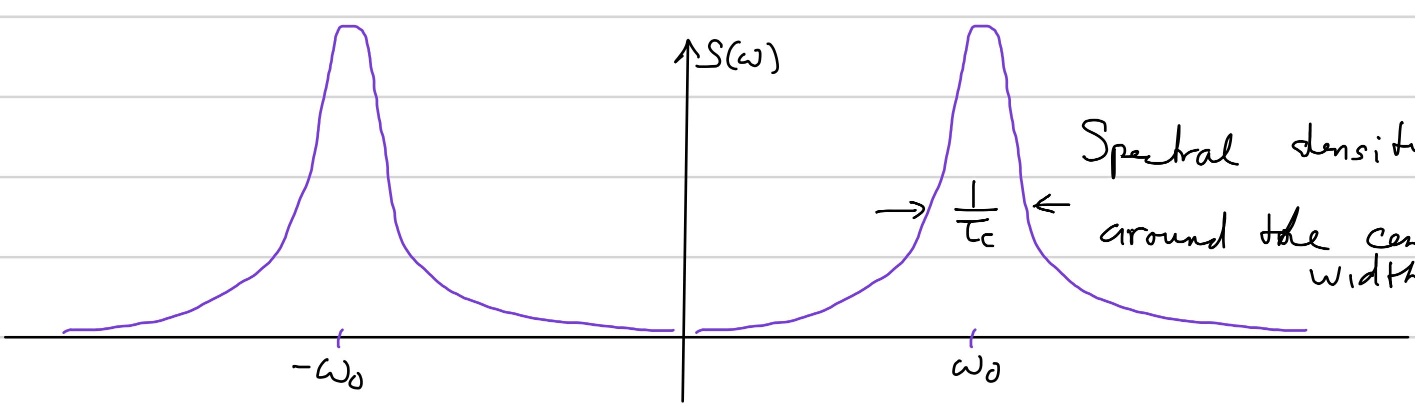
$$\langle \tilde{E}^*(0)\tilde{E}(\tau)\rangle = I_0\, e^{\left(-i\omega_0\tau - \frac{|\tau|}{\tau_c}\right)},$$ where ω_0 = central frequency, $\frac{1}{\tau_c}$ = Collision rate
I_0 = intensity at $\tau = 0$

$\Rightarrow$ Spectral density: $S(\omega) = \frac{I_0}{2}\int_{-\infty}^{\infty} d\tau\, \cos(\omega_0\tau)\, e^{\frac{-|\tau|}{\tau_c}}\, e^{+i\omega\tau} = \frac{I_0}{4}\int_{-\infty}^{\infty} d\tau\left[e^{+i(\omega-\omega_0)t - \frac{|\tau|}{\tau_c}} + e^{i(\omega+\omega_0)t - \frac{|\tau|}{\tau_c}}\right]$

Aside: $\int_{-\infty}^{\infty} d\tau\, e^{i\Omega\tau - \frac{|\tau|}{\tau_c}} = \int_0^\infty d\tau\, e^{\left(i\Omega - \frac{1}{\tau_c}\right)\tau} + \int_{-\infty}^{0} d\tau\, e^{\left(-i\Omega + \frac{1}{\tau_c}\right)\tau}$

$$= \frac{1}{-i\Omega - \frac{1}{\tau_c}} + \frac{-1}{-i\Omega + \frac{1}{\tau_c}} = \frac{2/\tau_c}{\Omega^2 + (1/\tau_c)^2}$$

$$\Rightarrow S(\omega) = \frac{I_0}{2}\left[\frac{1/\tau_c}{(\omega-\omega_0)^2 + (1/\tau_c)^2} + \frac{1/\tau_c}{(\omega+\omega_0) + (1/\tau_c)^2}\right]$$



Spectral density is Lorentzian around the central freq, width $\cong \frac{1}{\tau_c}$

The output intensity of the interferometer is the Fourier transform of the spectral density $\Rightarrow$ For a Lorentzian power spectral density, fringes decay exponentially. (See lecture notes).

Problem 2: Natural Light

The "natural light" emitted by stars or thermal lamp has a "finite coherence time". The amplitude and phase of the electric field of the wave are stochastic variables, specified by a probability distribution. One source of the randomness of the the field is collisions between the dipole emitters, which randomizes the phase and means the wave trains only have finite coherence lengths.

(a) Let $P_s(t)$ = "survival probabilty" = probability that a dipole oscillates for time t w/o a collision.

Let $\gamma = \frac{1}{\tau_0}$ = rate of collisions $\Rightarrow$ γdt = Probability of collision in infinitesimal dt

Under the "Markoff Approximation," the probability of a collision at any time t is completely independent of the "history" of the trajectory, i.e. the collision probability is uncorrelated between t and $t+dt$

Probability of surviving to $t+dt$ = (Prob. of surviving to t) $\times$ (Prob of no collision $t \to t+dt$)

$$P_s(t+dt) = \left(P_s(t) \right) \times \left(1 - \gamma dt \right)$$

$$\Rightarrow \boxed{\frac{1}{P_s}\frac{dP_s}{dt} = -\gamma \Rightarrow P_s(t) = e^{-\gamma t}}$$

(b) Let $p(t)dt$ = Probability of surviving ~~for~~ time a time t and then colliding in the interval $t \to t+dt$.

$$\Rightarrow \boxed{p(t)dt = \left(e^{-\gamma t}\right)\left(\gamma dt\right) = e^{-t/\tau_0}\frac{dt}{\tau_0}}$$

The collision cross section σ_0 define the rate of collisions, by definition:

$$\text{Rate of collision between particles} = \underbrace{n\bar{v}_{rel}}_{\text{flux of incident particles}}\sigma_0$$

flux of incident particles = density $\times$ rel. velocity.

In thermal equilibrium, each particle has a mean thermal speed $\overline{v^2} = \frac{3kT}{m}$ (equipartition) Maxwell - Boltzmann

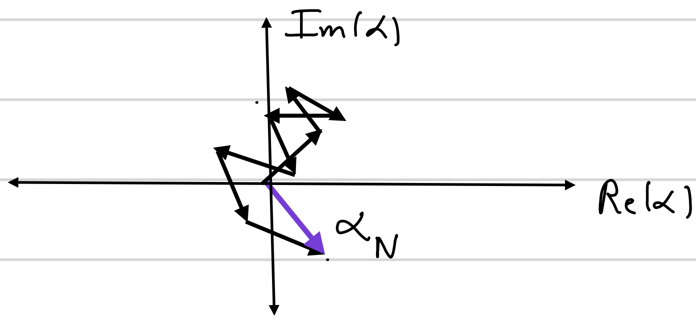
$$\Rightarrow \overline{v}_{rel}^2 = \langle(\vec{v}_1 - \vec{v}_2)^2\rangle = \langle v_1^2\rangle + \langle v_2^2\rangle = \frac{6k_BT}{m} \Rightarrow \boxed{\bar{v}_{rel} = \frac{6kT}{m}}$$

(c) The total electric field produced by a collection of N-oscillators with random phases:

$$E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t} e^{i\phi(t)} = E_0 e^{-i\omega t} \underbrace{\sum_{i=1}^{N} e^{i\phi(t)}}_{\alpha(t)} = E_0 e^{-i\omega t} a(t) e^{i\varphi(t)}$$

$\alpha(t) \leftarrow$ random complex number

The complex amplitude $\alpha(t)$ can be view as the end point of a random walk in 2D



A random walker has a Gaussian probability distribution of being away from the origin.

$$P(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)) = \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{[\mathrm{Re}(\alpha)]^2}{2\sigma^2}}\right]\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{[\mathrm{Im}(\alpha)]^2}{2\sigma^2}}\right]$$

Here the walker moves with fixed radius $\Rightarrow$ After N-steps $\langle|\alpha|^2\rangle = \langle(\mathrm{Re}(\alpha))^2\rangle + \langle(\mathrm{Im}(\alpha))^2\rangle$
$= 2\sigma^2 = N$

$$\Rightarrow \boxed{P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}}}$$

This probability distribution describes a field with random phase and mean zero amplitude.

(c) We seek the auto-correlation function at two different times for $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega_0 t} e^{i\phi_j(t)}$

$$\langle E^*(t) E(t+\tau)\rangle = \sum_i E_0^2 e^{-i\omega_0\tau} \langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\rangle + \sum_{i\neq j} E_0^2 e^{-i\omega_0\tau} \langle e^{-i\phi_i(t)} e^{+i\phi_j(t+\tau)}\rangle$$

Because the oscillators are uncorrelated $\langle e^{-i\phi_i(t)} e^{+i\phi_j(t+\tau)}\rangle_{i\neq j} = \langle e^{-i\phi_i(t)}\rangle\langle e^{i\phi_j(t+\tau)}\rangle = 0$

Aside: $\langle e^{-i(\phi_i(t+\tau)-\phi_i(t))}\rangle = 0$ unless the oscillator does not collide in time τ.

$= 1 -$ Probality that the oscillator collides in time interval τ

$= 1 - \int_0^\tau p(t)\,dt = e^{-\tau/\tau_0}$ (the same for all oscillators)

$$\boxed{\Rightarrow \langle E^*(t) E(t+\tau)\rangle = N E_0^2 e^{-i\omega_0 t} e^{-\tau/\tau_0}}$$

(d) While the mean electric field of natural light is zero, there are fluctuations. This implies that there is an average intensity and fluctuations in intensity

$$\boxed{\langle I(t)\rangle = \langle E^*(t)\,E(t)\rangle \underset{\substack{\uparrow \\ \text{from part (c)}}}{=} N E_0^2 \ \text{(intensities add for incoherent oscillators).}}$$

Generally, we can derive the probability distribution $P(I(t))$ using $P(\alpha(t))$:

$$I(t) = |E(t)|^2 = E_0^2 |\alpha(t)|^2 \Rightarrow P(I(t))\,dI = P(I(\alpha(t)))\,d^2\alpha = P(|\alpha|)\,2\pi|\alpha|\,d|\alpha|$$

$$\Rightarrow P(I(t)) = 2\pi|\alpha|\left(\frac{dI}{d|\alpha|}\right)^{-1} P(|\alpha|) = 2\pi|\alpha|\left(2E_0^2|\alpha(t)|\right)^{-1}\frac{1}{N\pi}\,e^{-\frac{|\alpha|^2}{N}} = \frac{1}{NE_0^2}\,e^{-\frac{E_0^2|\alpha(t)|^2}{E_0^2 N}}$$

$$\Rightarrow \boxed{P(I(t)) = \frac{1}{\langle I\rangle}\,e^{-\frac{I(t)}{\langle I\rangle}}}$$

We can calculate the moments of this distribution:

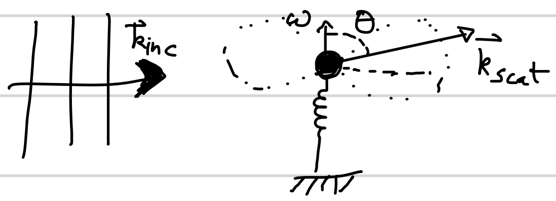
$$\boxed{\langle I^n\rangle = \int_0^\infty I^n\,\frac{e^{-I/\langle I\rangle}}{\langle I\rangle}\,dI = n!\,\langle I\rangle^n}$$

In particular $\langle I^2\rangle = 2\langle I\rangle^2 \Rightarrow \Delta I^2 = \langle I^2\rangle - \langle I\rangle^2 = \boxed{\langle I\rangle^2 \Rightarrow \Delta I = \langle I\rangle}$

For natural light, the fluctuations in intensity are equal to the mean.

Problem 3

We consider scattering of an electromagnetic wave by a classical electromagnetic wave.



(a) The absorption cross-section is defined: $P_{abs} = \sigma_{abs} I_{inc}$

The incident intensity $I_{inc} = \frac{c}{8\pi} E_0^2$

The absorbed power = Rate at which work in done on oscillator $= \omega \, \mathrm{Im}(\tilde{\alpha}(\omega)) \frac{E_0^2}{2}$

where $\tilde{\alpha}(\omega)$ = dipole polarizability, $\tilde{\alpha}(\omega) = \frac{e^2/2m\omega}{-\Delta - i\Gamma/2} \Rightarrow \mathrm{Im}(\tilde{\alpha}(\omega)) = \frac{e^2}{4m\omega} \frac{\Gamma}{\Delta^2 + \Gamma^2/4}$

$$\Rightarrow \boxed{\sigma_{abs} = \frac{2\pi e^2}{mc} \frac{\Gamma/2}{\Delta^2 + \frac{\Gamma^2}{4}} = \frac{2\pi^2 e^2}{mc} g(\omega), \quad \text{where} \quad g(\omega) = \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + \Gamma^2/4} \quad \left(\begin{array}{c}\text{Natural}\\ \text{lineshape}\end{array}\right)}$$

(b) We seek the differential scattering cross section. By definition,

$$dP_{scat} = I_{inc}\, d\sigma_{scat} = \frac{dP_{scat}}{d\Omega} d\Omega = \frac{dP_{rad}}{d\Omega} d\Omega$$ since the scattered power = power radiated

Now $dP_{rad} = I_{rad}(r^2 d\Omega)$, where $I_{rad} = \frac{c}{8\pi} E_{rad}^2$ is the radiated intensity into solid angle $d\Omega$

From dipole radiation theory, $|\vec{E}_{rad}| = \left(\frac{\omega}{c}\right)^2 |\tilde{\alpha}| E_0 \sin\theta \left|\frac{e^{ikr}}{r}\right|$

$$\Rightarrow dP_{rad} = \frac{\omega^4}{8\pi c^3} |\tilde{\alpha}|^2 E_0^2 \sin^2\theta \, d\Omega$$

Putting all of this together: $\boxed{\frac{d\sigma_{scat}}{d\Omega} = \frac{\omega^4}{c^4} |\tilde{\alpha}|^2 \sin^2\theta}$

Integrating over all solid angles, $\sigma_{scat} = \int \frac{d\sigma_{scat}}{d\Omega} d\Omega = \int \frac{d\sigma_{scat}}{d\Omega} 2\pi \, d(\cos\theta)$

Let $\mu = \cos\theta$ $\Rightarrow \boxed{\sigma_{scat} = 2\pi \frac{\omega^4}{c^4} |\tilde{\alpha}| \int_{-1}^{1} (1-\mu^2)\, d\mu = \frac{8\pi}{3} \frac{\omega^4}{c^4} |\tilde{\alpha}|^2 = \frac{2\pi}{3} \frac{e^4}{m^2c^4} \frac{\omega^2}{\Delta^2 + \frac{\Gamma^2}{4}}}$

If the damping is radiative only, $\sigma_{abs} = \sigma_{scat}$, i.e., all absorbed light is scattered

$$\Rightarrow \frac{\pi e^2}{mc} \frac{\Gamma_{rad}}{\Delta^2 + \Gamma_{rad}^2/4} = \frac{2\pi}{3} \frac{e^4}{m^2 c^4} \frac{\omega^2}{\Delta^2 + \Gamma^2/4} \Rightarrow \boxed{\Gamma_{rad} = \frac{2}{3} \frac{e^2}{mc^3} \omega^2}$$

Note: I have been pretty cavalier, switching back and forth between ω & ω_0 in some of the definitions. From the physics described here, the rate of energy decay will depend on the acceleration, and thus the driving frequency ω, not ω_0. However, the natural linewidth quantum mechanically is a property of the transition, and thus depends only on ω_0. The classical theory of radiation reaction cannot capture these small differences, and so you will often see ratios $\left(\frac{\omega}{\omega_0}\right)^2$ appearing which we set to 1.

(c) An equivalent approach: Equate the absorbed power to the total radiated power

Time-averaged

$$P_{abs} = \omega \, \mathrm{Im}(\tilde{\alpha}(\omega)) \frac{E_0^2}{2} = \frac{e^2}{4m} \frac{\Gamma_{rad}/2}{\Delta^2 + \Gamma_{rad}^2/4} E_0^2$$

$$P_{rad} = \frac{1}{3} \frac{e^2}{c^3} |\ddot{\vec{r}}|^2 = \frac{1}{3} \frac{|\ddot{\vec{d}}|^2}{c^3} = \frac{\omega^4}{3c^3} |\tilde{\alpha}|^2 E_0^2 = \frac{1}{3} \frac{e^4 \omega^2}{4m^2 c^3} \frac{1}{\Delta^2 + \Gamma_{rad}^2/4} E_0^2$$

Equating $P_{abs} = P_{rad} \Rightarrow \Gamma_{rad} = \frac{2}{3} \frac{e^2 \omega^2}{mc^3} = \frac{2}{3}(k r_c)\omega$

For the case of the sodium D2 resonance, $\lambda = 589\,\mathrm{nm} \Rightarrow \Gamma_{rad} = \frac{2}{3}\left(\frac{2\pi r_c}{\lambda}\right)\left(\frac{2\pi c}{\lambda}\right)$

With the classical electron radius $r_c = 2.8 \times 10^{-15}\,\mathrm{m} \Rightarrow$ $\boxed{\frac{\Gamma_{rad}}{2\pi} \cong 10\,\mathrm{MHz}}$ very close! to quantum 9.8 MHz

The oscillator strength: $\boxed{f = \frac{\Gamma_{quant}}{\Gamma_{class}} = 0.98}$

(d) We found the total scattering cross section $\sigma_{scat} = \frac{2\pi}{3} \frac{e^4}{m^2 c^4} \frac{\omega^2}{\Delta^2 + \frac{\Gamma^2}{4}} = \frac{\pi e^2}{mc} \frac{\Gamma_{rad}}{\Delta^2 + \Gamma_{rad}^2/4}$

$$\Rightarrow \sigma_{scat} = \frac{3\pi}{2} \frac{c^2}{\omega^2} \frac{\Gamma_{rad}^2}{\Delta^2 + \Gamma^2/4} \Rightarrow \boxed{\sigma_{scat} = 6\pi \lambdabar^2 \frac{1}{1 + \frac{4\Delta^2}{\Gamma_{rad}^2}}} \qquad \lambdabar \equiv \frac{\lambda}{2\pi}$$

The resonant cross section is $6\pi\lambdabar^2$. The same is true quantum mechanically.